RECOGNISING ACHIEVEMENT

ADVANCED GCE **4758/01**

MATHEMATICS (MEI)

Differential Equations

THURSDAY 24 JANUARY 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

1 The differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = f(t)$ is to be solved for $t \geqslant 0$ subject to the conditions that $\frac{dy}{dt} = 0$ and $y = 0$ when $t = 0$.

Firstly consider the case $f(t) = 2$.

(i) Find the solution for y in terms of t. [10]

Now consider the case $f(t) = e^{-t}$.

(ii) Explain briefly why a particular integral cannot be of the form ae^{-t} or ate^{-t}. Find a particular integral and hence solve the differential equation, subject to the given conditions. [8]

(iii) For $t > 0$, show that $y > 0$ and find the maximum value of y. Hence sketch the solution for $t \geqslant 0$. [You may assume that $t^k e^{-t} \to 0$ as $t \to \infty$ for any k.] [6]

2 A raindrop falls from rest through mist. Its velocity, $v\,\mathrm{m\,s^{-1}}$ vertically downwards, at time t seconds after it starts to fall is modelled by the differential equation

$$(1+t)\frac{dv}{dt} + 3v = (1+t)g - 3.$$

(i) Solve the differential equation to show that $v = \frac{1}{4}g(1+t) - 1 + (1 - \frac{1}{4}g)(1+t)^{-3}$. [10]

The model is refined and the term -3 is replaced by the term $-2v$, giving the differential equation

$$(1+t)\frac{dv}{dt} + 3v = (1+t)g - 2v.$$

(ii) Find the solution subject to the same initial conditions as before. [9]

(iii) For each model, describe what happens to the acceleration of the raindrop as $t \to \infty$. [5]

3 The population, P, of a species at time t years is to be modelled by a differential equation. The initial population is 2000.

At first the model $\frac{dP}{dt} = 0.5P$ is used.

(i) Find P in terms of t. [3]

To take account of observed fluctuations, the model is refined to give $\frac{dP}{dt} = 0.5P + 170 \sin 2t$.

(ii) State the complementary function for this differential equation. Find a particular integral and hence state the general solution. [8]

(iii) Find the solution subject to the given initial condition. [2]

The model is further refined to give $\frac{dP}{dt} = 0.5P + P^{\frac{2}{3}} \sin 2t$. This is to be solved using Euler's method. The algorithm is given by $t_{r+1} = t_r + h$, $P_{r+1} = P_r + h\dot{P}_r$.

(iv) Using a step length of 0.1 and the given initial conditions, perform two iterations of the algorithm to estimate the population when $t = 0.2$. [4]

The population is observed to tend to a non-zero finite limit as $t \to \infty$, so a further model is proposed, given by

$$\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}}.$$

(v) Without solving the differential equation,

(A) find the limiting value of P as $t \to \infty$, [3]

(B) find the value of P for which the rate of population growth is greatest. [4]

4 The simultaneous differential equations

$$\frac{dx}{dt} = -3x + y + 9,$$
$$\frac{dy}{dt} = -5x + y + 15,$$

are to be solved for $t \geqslant 0$.

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 6$. [5]

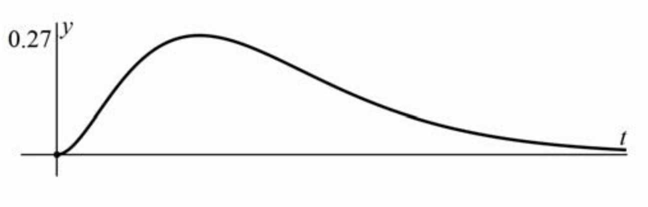
(ii) Find the general solution for x. [7]

(iii) Hence find the corresponding general solution for y. [3]

(iv) Find the solutions subject to the conditions that $x = y = 0$ when $t = 0$. [4]

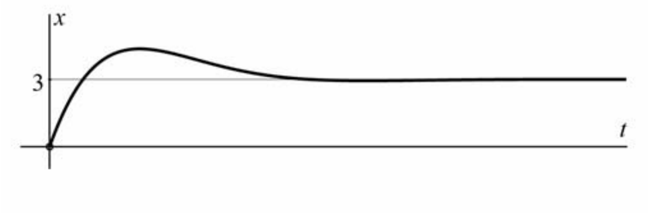
(v) Sketch, on separate axes, graphs of the solutions for $t \geqslant 0$. [5]

4758 Differential Equations

1(i)	$\alpha^2+2\alpha+1=0$	M1	Auxiliary equation	
	$\alpha=-1$ (repeated)	A1		
	CF $y=(A+Bt)e^{-t}$	F1	CF for their roots	
	PI $y=a$	B1	Constant PI	
	in DE $\Rightarrow y=2$	B1	PI correct	
	$y=2+(A+Bt)e^{-t}$	F1	Their PI + CF (with two arbitrary constants)	
	$t=0, y=0 \Rightarrow 0=2+A \Rightarrow A=-2$	M1	Condition on y	
	$\dot{y}=(B-A-Bt)e^{-t}$	M1	Differentiate (product rule)	
	$t=0, \dot{y}=0 \Rightarrow 0=B-A \Rightarrow B=-2$	M1	Condition on $\dot{y}$	
	$y=2-(2+2t)e^{-t}$	A1		10
(ii)	Both terms in CF hence will give zero if substituted in LHS	E1		
	PI $y=bt^2 e^{-t}$	B1		
	$\dot{y}=(2bt-bt^2)e^{-t}, \ddot{y}=(2b-4bt+bt^2)e^{-t}$			
	in DE $\Rightarrow (2b-4bt+bt^2+2(2bt-bt^2)+bt^2)e^{-t}=e^{-t}$	M1	Differentiate twice and substitute	
	$\Rightarrow b=\frac{1}{2}$	A1	PI correct	
	$y=(C+Dt+\frac{1}{2}t^2)e^{-t}$	F1	Their PI + CF (with two arbitrary constants)	
	$t=0, y=0 \Rightarrow 0=C$	M1	Condition on y	
	$\dot{y}=(D+t-C-Dt-\frac{1}{2}t^2)e^{-t}$			
	$t=0, \dot{y}=0 \Rightarrow 0=D-C \Rightarrow D=0$	M1	Condition on $\dot{y}$	
	$y=\frac{1}{2}t^2 e^{-t}$	A1		8
(iii)	$t>0 \Rightarrow \frac{1}{2}t^2>0$ and $e^{-t}>0 \Rightarrow y>0$	E1		
	$\dot{y}=(t-\frac{1}{2}t^2)e^{-t}$ so $\dot{y}=0 \Leftrightarrow t-\frac{1}{2}t^2=0 \Leftrightarrow t=0$ or 2	M1	Solve $\dot{y}=0$	
	Maximum at $t=2, y=2e^{-2}$	A1	Maximum value of y	
		B1 B1 B1	Starts at origin Maximum at their value of y $y>0$	6

2(i)	$\frac{dv}{dt}+\frac{3}{1+t}v=g-\frac{3}{1+t}$	M1	Rearrange	
	$I=\exp\left(\int\frac{3}{1+t}\,dt\right)=e^{3\ln(1+t)}=(1+t)^3$	M1 A1 A1	Attempt integrating factor Correct Simplified	
	$(1+t)^3\frac{dv}{dt}+3(1+t)^2v=g(1+t)^3-3(1+t)^2$	F1	Multiply DE by their *I*	
	$\frac{d}{dt}\left((1+t)^3v\right)=g(1+t)^3-3(1+t)^2$			
	$(1+t)^3v=\int\left(g(1+t)^3-3(1+t)^2\right)dx$	M1	Integrate	
	$=\frac{1}{4}g(1+t)^4-(1+t)^3+A$	A1	RHS	
	$v=\frac{1}{4}g(1+t)-1+A(1+t)^{-3}$	F1	Divide by their *I* (must also divide constant)	
	$t=0,v=0\Rightarrow 0=\frac{1}{4}g-1+A$	M1	Use condition	
	$v=\frac{1}{4}g(1+t)-1+\left(1-\frac{1}{4}g\right)(1+t)^{-3}$	E1	Convincingly shown	
				10
(ii)	$(1+t)\frac{dv}{dt}+5v=(1+t)g$	M1	Rearrange	
	$\frac{dv}{dt}+\frac{5}{1+t}v=g$			
	$I=\exp\left(\int\frac{5}{1+t}\,dt\right)=e^{5\ln(1+t)}=(1+t)^5$	M1 A1	Attempt integrating factor Simplified	
	$(1+t)^5\frac{dv}{dt}+5(1+t)^4v=g(1+t)^5$	F1	Multiply DE by their *I*	
	$\frac{d}{dt}\left((1+t)^5v\right)=g(1+t)^5$			
	$(1+t)^5v=\int g(1+t)^5\,dx$	M1	Integrate	
	$=\frac{1}{6}g(1+t)^6+B$	A1	RHS	
	$v=\frac{1}{6}g(1+t)+B(1+t)^{-5}$	F1	Divide by their *I* (must also divide constant)	
	$t=0,v=0\Rightarrow 0=\frac{1}{6}g+B$	M1	Use condition	
	$v=\frac{1}{6}g\left(1+t-(1+t)^{-5}\right)$	F1	Follow a non-trivial GS	
				9
(iii)	First model: $\frac{dv}{dt}=\frac{1}{4}g-3\left(1-\frac{1}{4}g\right)(1+t)^{-4}$	M1	Find acceleration	
	As $t\to\infty,(1+t)^{-4}\to 0$	B1	Identify term(s) → 0 in their solution for either model	
	Hence acceleration tends to $\frac{1}{4}g$	A1		
	Second model $\frac{dv}{dt}=\frac{1}{6}g\left(1+5(1+t)^{-6}\right)$	M1	Find acceleration	
	Hence acceleration tends to $\frac{1}{6}g$	A1		
				5

3(i)	$P = Ae^{0.5t}$	M1	Any valid method	
	$t = 0, P = 2000 \Rightarrow A = 2000$	M1	Use condition	
	$P = 2000e^{0.5t}$	A1		3
(ii)	CF $P = Ae^{0.5t}$	F1	Correct or follows (i)	
	PI $P = a\cos 2t + b\sin 2t$	B1		
	$\dot{P} = -2a\sin 2t + 2b\cos 2t$	M1	Differentiate	
	$-2a\sin 2t + 2b\cos 2t = 0.5(a\cos 2t + b\sin 2t) + 170\sin 2t$	M1	Substitute	
	$-2a = 0.5b + 170$	M1	Compare coefficients	
	$2b = 0.5a$	M1	Solve	
	solving $\Rightarrow a = -80, b = -20$	A1		
	GS $P = Ae^{0.5t} - 80\cos 2t - 20\sin 2t$	F1	Their PI + CF (with one arbitrary constant)	8
(iii)	$t = 0, P = 2000 \Rightarrow A = 2080$	M1	Use condition	
	$P = 2080e^{0.5t} - 80\cos 2t - 20\sin 2t$	F1	Follow a non-trivial GS	2
(iv)	t P $\dot{P}$	M1	Use of algorithm	
	0 2000 1000	A1	2100	
	0.1 2100 1082.58	A1	1082.5…	
	0.2 2208	A1	2208	4
(v)	*(A)* Limiting value $\Rightarrow \dot{P} = 0$	M1	Set $\dot{P} = 0$	
	$\Rightarrow P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}} = 0$	M1	Solve	
	(as limit non-zero) limiting value = 12000	A1		3
	(B) Growth rate max when			
	$f(P) = P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}}$ max	M1	Recognise expression to maximise	
	$f'(P) = \left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}} - \frac{1}{2 \times 12000}P\left(1 - \frac{P}{12\,000}\right)^{-\frac{1}{2}}$	M1	Reasonable attempt at derivative	
	$f'(P) = 0 \Leftrightarrow \left(1 - \frac{P}{12\,000}\right) - \frac{1}{2 \times 12000}P = 0$	M1	Set derivative to zero	
	$\Leftrightarrow P = 8000$	A1		4

4(i)	$\ddot{x} = -3\dot{x} + \dot{y}$	M1	Differentiate first equation	
	$= -3\dot{x} + (-5x + y + 15)$	M1	Substitute for $\dot{y}$	
	$y = 3x - 9 + \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = -3\dot{x} - 5x + (3x - 9 + \dot{x}) + 15$	M1	Substitute for y	
	$\ddot{x} + 2\dot{x} + 2x = 6$	E1		
				5
(ii)	$\lambda^2 + 2\lambda + 2 = 0$	M1	Auxiliary equation	
	$\lambda = -1 \pm j$	A1		
	CF $x = e^{-t}(A\cos t + B\sin t)$	M1	CF for complex roots	
		F1	CF for their roots	
	PI $x = a$	B1	Constant PI	
	$2a = 6 \Rightarrow a = 3$	B1	PI correct	
	GS $x = 3 + e^{-t}(A\cos t + B\sin t)$	F1	Their CF + PI (with two arbitrary constants)	
				7
(iii)	$y = 3x - 9 + \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$= 9 + 3e^{-t}(A\cos t + B\sin t) - 9$ $- e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t)$	M1	Differentiate x and substitute	
	$y = e^{-t}((2A + B)\cos t + (2B - A)\sin t)$	A1	Constants must correspond with those in x	
				3
(iv)	$0 = 3 + A \Rightarrow A = -3$	M1	Condition on x	
	$0 = 2A + B \Rightarrow B = 6$	M1	Condition on y	
	$x = 3 + 3e^{-t}(2\sin t - \cos t)$	F1	Follow their GS	
	$y = 15e^{-t}\sin t$	F1	Follow their GS	
				4
(v)		B1	Sketch of x starts at origin	
		B1	Asymptote $x = 3$	
		B1	Sketch of y starts at origin	
		B1	Decaying oscillations (may decay rapidly)	
		B1	Asymptote $y = 0$	
				5

4758: Differential Equations (Written paper)

General Comments

Many candidates demonstrated a good understanding of the specification and high levels of algebraic competency. Questions 1 and 4 proved to be the most popular choices.
When sketching graphs, if candidates use a graphic calculator, merely copying the screen may not be enough if it does not identify the key features of the solution. However, detailed analysis is not required in sketch graphs unless specifically requested in the question.
When a differential equation and conditions are given, a request to find the solution implies that the conditions should be used. (ie in these circumstances, find the particular solution unless the specific term 'general solution' is used.)

Comments on Individual Questions

Section A

1) (i) This was generally done very well, although a few candidates did not use the given conditions to calculate the arbitrary constants.

(ii) Although there were some excellent answers, most candidates struggled to explain why the given expressions could not form a particular integral. They were required to say more than just identify them as terms in the complementary function, but to remark that they would give zero in the left hand side of the differential equation.

(iii) This was often done well, although some candidates found the value of t at the maximum, rather than the value of y. It was expected that the maximum value was marked on the sketch graph.

2) (i) This was often done very well, although some candidates made errors when dividing or multiplying the equation through by an appropriate expression. For example, when dividing through by (1 + t), some candidates did not divide the –3 term.

(ii) This was also often done well, although some candidates wrongly used the same integrating factor as before.

(iii) Completely correct answers were not common. Many candidates made errors in differentiating their velocity expressions. A few described the velocity rather than the acceleration.

3) (i) A surprising number of candidates did not use the condition to calculate the arbitrary constant. Note that the question did not ask for the general solution and so the condition should have been used.

(ii) This was often done well, although algebraic errors when calculating the coefficients were common.

(iii) The particular solution was usually well answered.

(iv) The numerical solution was often done well, although a sizeable minority made numerical errors in the second step.

(v) Finding the limiting value was sometimes well done, although many did not realise that this must correspond with a zero derivative. Finding the population when the growth rate is greatest was rarely completed. A number of candidates differentiated the expression, but few were able to solve the resulting equation.

4) (i) Most candidates completed this correctly, but a few did not seem to know how to do the elimination.

(ii) This was often correct, but some candidates assumed that the particular integral was 6.

(iii) Many candidates gave correct answers by using their solution for x in the first of the displayed differential equations. Pleasingly few candidates attempted to construct a differential equation for y.

(iv) Many correct solutions were seen.

(v) The sketches were often done well, but some candidates omitted to identify the key features, in particular the initial conditions and the asymptotes.